AN ALTERNATIVE APPROACH TO FUZZY CONTROL OF
A CLASS OF NONLINEAR SYSTEMS

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ABSTRACT
This paper proposes an alternative approach to fuzzy control of
a class of nonlinear systems. In this study, we consider a
nonlocal approach. The linear feedback techniques are employed
to design the controller incorporating the optimal control
principle. In addition, the design procedure is presented.
Eventually, an example of the balancing and swing-up of an
inverted pendulum on a car is given to show the result of the
present paper.

1. INTRODUCTION
Stability is one of the most important issues in the analysis and
design of control systems. Stability analysis of fuzzy control
systems has been difficult because these systems are essentially
coupled. Recently, some stability results [4, 5] which are
based on linear stability theory have been reported. The
stability criterion is reduced to a problem of finding a common
lyapunov function for a set of lyapunov inequalities.
In this paper, we consider a nonlocal approach. We can use the
linear feedback control techniques in the case of feedback
stabilization. The feedback control law is designed by the
optimal control technique. The procedure is given as follows:
First the nonlinear plant is represented by a Takagi-Sugeno type
fuzzy model. In this type of fuzzy model, local dynamics in
different state space regions are represented by linear models.
The overall model of the system is achieved by using a fuzzy
“blending” of these linear models. The control design is carried
out based on the fuzzy model via the so-called parallel
distributed compensation scheme. The idea is that for each local
model, a linear optimal feedback control is designed. The
resulting overall controller, which is nonlinear in general, is
again a fuzzy blending of each individual linear optimal
controller.
This paper deals with the stability and design issues in the
proposed fuzzy control approach of nonlinear systems. More
significantly, the stability analysis and control design problems
are reduced to linear matrix inequality (LMI) problems[?].
Numerically, the LMI problems can be solved very efficiently by
means of some of the most powerful tools available to date in
the mathematical programming literature. Therefore, recasting
the stability analysis and control design problems as LMI
problems is equivalent to finding solutions to the original
problems.

2. THE PROBLEM FORMULATION
Consider a nonlinear system which is represented in a
continuous fuzzy dynamic model [1]. The continuous fuzzy
dynamic model, which is called the "Takagi-Sugeno fuzzy
model", represents local linear input-output relations of
nonlinear systems. The ith rule of this fuzzy model is of the
following form:
Plant Rule i:
If x_i(t) is M_i and... and x_j(t) is M_j
then x(t) = A_i x(t) + B_i u(t), i = 1,..., r
(1)
where M_i is the fuzzy set, x(t) ∈ R^n is the state vector,
u(t) ∈ R^m is the input vector, A_i ∈ R^{n×n}, and B_i ∈ R^{n×m}, r
is the number of if-then rules. x_i(t) are the variables of the
antecedent part.
If a pair of (x_i(t), u_i(t)) is given, then the final output of the
fuzzy system is inferred as follows:
\[ x(t) = \sum_{i=1}^{r} \sum_{j=1}^{p} w_i(x_i(t)) M_j(s_j(t)) \]
where w_i(x_i(t)) = \frac{1}{\sum_{j=1}^{p} M_j(s_j(t))} M_i(x_i(t)) is the grade of
membership of x_i(t) in M_i. We assumed that w_i(x_i(t)) ≥ 0 for
i = 1,..., r and \sum_{i=1}^{r} w_i(x_i(t)) > 0 for all t.
The open-loop systems of (2) are defined as
\[ \dot{x}(t) = \sum_{i=1}^{n} \frac{w_i(x(t))}{\sum_{i=1}^{n} w_i(x(t))} A_i x(t) \]  
(3)

The concept of parallel distributed compensation (PDC) is shown in Fig. 1. For each rule, we can use linear feedback control design techniques. The fuzzy controller which shares the same fuzzy sets with the fuzzy system (2), is as the following form.

Control Rule 1:
If \( x_i(t) \) is \( M_i \) and \( x_j(t) \) is \( M_j \)
then \( u(t) = -K_j x(t) \)
where \( i = 1, 2, ..., r \)

Rule 2:
Rule 3:
[Linear controller design technique]

Fig.1 Parallel distributed compensation (PDC) design

Hence, the inferred output of the fuzzy controller is
\[ u(t) = \frac{\sum_{i=1}^{r} w_i(x(t)) K_i x(t)}{\sum_{j=1}^{n} w_j(x(t))} \]  
(4)

Note that the controller (4) is nonlinear in general. Substituting (4) to (2) we obtain
\[ \dot{x}(t) = \frac{\sum_{i=1}^{r} w_i(x(t)) w_j(x(t)) (A_i - B_i K_j) x(t)}{\sum_{j=1}^{n} w_j(x(t))} \]  
(5)

The main control objective is to design a controller to stabilize the system. Hence, we should evaluate the linear feedback gain for each fuzzy control rule to make the overall system stable.

3. THE CONTROLLER DESIGN PROCEDURE

In this section, we will propose the design procedure. First, we separate the state space into several subspaces. Then we evaluate the linearized system matrices about the operating point of these subspaces. Finally, the consequent part of the Takagi-Sugeno fuzzy model of each rule is obtained. Assume that the pair of \( \{A, B\} \) is controllable for \( i = 1, 2, ..., r \).

Then we can define a performance index \( J \) of this nonlinear system.
\[ J = \frac{1}{2} \left( x(t)^T Q x(t) + u(t)^T R u(t) \right) \]  
(6)

where \( Q \) and \( R \) can be chosen according to the constraints and requirements.

There is an additional assumption that the pair of \( \{A, B\} \) is detectable, where \( D \) is any matrix such that \( D P = Q \).

Next, the concept of PDC is utilized. For each linearized model of the Takagi-Sugeno fuzzy rules, we design the controllers \( K_i \) \( i = 1, 2, ..., r \) which will minimize \( J \). Hence, we need to solve the following Riccati equations.
\[ \dot{P}_i + A_i^T P_i - P_i B_i R_i^{-1} B_i^T P_i + Q_i = 0 \]  
for \( i = 1, 2, ..., r \)  
(7)

Then, for each fuzzy rule, the linear optimal feedback control is obtained as 
\[ u_i = -R_i^{-1} B_i^T P_i x \]  
for \( i = 1, 2, ..., r \)  
(8)

Although the linear feedback gains of the fuzzy rules are designed, we don't know whether the overall system is stable. So the stability conditions of the open-loop fuzzy system are given below.

Theorem 1: The equilibrium point of (3) is asymptotically stable in the large if there exists a common positive definite matrix \( P \) such that
\[ A_i^T P + PA_i < 0 \]  
(9)

for \( i = 1, 2, ..., r \).

We apply Theorem 1 to the closed-loop fuzzy system. Then we obtain the following theorem.

Theorem 2: The equilibrium point of a fuzzy control system (5) is asymptotically stable in the large if there exists a common positive definite matrix \( P \) such that
\[ (A_i - B_i K_j)^T P + P (A_i - B_i K_j) < 0 \]  
for \( i, j = 1, 2, ..., r \)  
(10)

We can observe that (9) depends only on \( A_i \), not on \( w_j(x(t)) \). However, we should note that Theorem 1 just gives a sufficient condition for ensuring stability of (3).

To check the stability, we need to find a common positive definite matrix \( P \) which satisfies the sufficient condition (10). We can use the LMI problem to determine if such a \( P \) exists. This is an LMI problem. Numerically, the LMI problem can be solved very efficiently by means of some of the most powerful both available to date in mathematical programming literature [2]. For instance, the recently developed interior-point methods [6] are extremely efficient in practice.

Hence, we can use the LMIs to solve the positive definite matrix \( P \) or determine that no such \( P \) exists. If the stability conditions are not satisfied, we have to repeat the procedure by giving another new pair of \( Q \) and \( R \).

The proposed method in this paper is different from that of [8]. The states of the systems and control input are both considered. According to the control objectives, some hardware constraints and control input constraints, we are able to choose the adequate pair of \( Q \) and \( R \). Hence, this is a more practical case. In addition, the linear optimal feedback gains can be easily evaluated by the "MATLAB" tool.

4. EXAMPLE

We will consider the problem of balancing and swing-up of an inverted pendulum on a cart to illustrate the PDC approach. The equations of motion for the pendulum are formulated as follows [7].
\[ \dot{x}_t = x_r \]  
\[ \dot{x}_r = g \sin(x_r) - \frac{\sin(2x_r)}{4} - \cos(x_r) \]  
(11)

where \( x_t \) denotes the angle (in radians) of the pendulum from the vertical, and \( x_r \) is the angular velocity. \( g = 9.8 \text{ms}^{-2} \) is the gravity constant, \( m \) is the mass of pendulum, \( M \) is the mass of the cart, \( 2l \) is the length of the pendulum, and \( Q = \frac{mg}{4} \) is the force applied to the cart (in Newtons). We choose \( m = 2.0 \text{kg}, M = 8.0 \text{kg}, 2l = 1.0 \text{m} \) in the simulations.
The control objective is to balance the inverted pendulum for the approximate range $x_c(-\pi/2, \pi/2)$. First, we must have a fuzzy model which represents the dynamics of the nonlinear plant. We approximate the system by the following two-rule fuzzy model.

**Plant rules**

**Rule 1**: If $x_1$ is about 0,

$$
\begin{bmatrix}
0 & 1 \\
1.729 & 0
\end{bmatrix}
\begin{bmatrix}
x_1 \\
x_2
\end{bmatrix} +
\begin{bmatrix}
0 \\
-0.1765
\end{bmatrix}
= A_1 x + B_1 u
$$

**Rule 2**: If $x_1$ is about $\pm \pi/2$ ($|x_1| \leq \pi/2$),

$$
\begin{bmatrix}
0 & 1 \\
9.36 & 0
\end{bmatrix}
\begin{bmatrix}
x_1 \\
x_2
\end{bmatrix} +
\begin{bmatrix}
0 \\
-0.0052
\end{bmatrix}
= A_2 x + B_2 u
$$

**Membership functions for Rule 1 and 2 are shown in Fig. 2.**

![Membership functions of two-rule model](image)

Fig. 2 Membership functions of two-rule model

We will choose the two cases of $(Q, R)$, $\begin{bmatrix} 1 & 0 \\ 0 & 100 \end{bmatrix}$ and $\begin{bmatrix} 1000 & 0 \\ 0 & 1000 \end{bmatrix}$. Then we can find out the common positive definite matrix $\begin{bmatrix} 1.804 & 1.897 \\ 1.897 & 1.804 \end{bmatrix}$ for case 1 and $\begin{bmatrix} 1.259 & 3.0715 \\ 3.0715 & 1.259 \end{bmatrix}$ for case 2. By the **Theorem 2**, the stability of the fuzzy control system (fuzzy model + PDC control) for these two cases can be guaranteed. Fig. 3 and Fig. 4 show the response with the fuzzy controller for initial conditions $x_1 = 30^\circ, 45^\circ, 60^\circ, 88^\circ$, and $x_2 = 0$, for these two cases, respectively.

![Angle response for different initial conditions](image)

Fig. 3 The angle response for different initial conditions choosing $Q = \text{diag}(1,100)$ and $R = 1$

![Angle response for different initial conditions](image)

Fig. 4 The angle response for different initial conditions choosing $Q = \text{diag}(10000,10000)$ and $R = 1$

From the above results, we can find out that different $Q$ and $R$ will yield different responses. In case 1, because the weight of state $x_1$ is much bigger than others, the state $x_1$ will achieve a nonzero final value. In case 2, $Q$ is $10^4$ times of $R$, due to the more attentions on the state $x_2$ than on the control input. Consequently, the final angle of the inverted pendulum will converge to a value approximately closer to zero. Therefore, we can choose the appropriate pair of $Q$ and $R$ according to the control objective and the control input constraints.

5. CONCLUSION

Stability conditions of fuzzy models and fuzzy control systems has been given. A design methodology for stabilization of a class of nonlinear systems based on Takagi-Sugeno fuzzy model, PDC control design, and linear quadratic optimal control is presented. The design procedure is conceptually simple and systematic. Moreover, the stability analysis and control design problems are reduced to LMIs problems. Therefore, they can be solved very efficiently in practice by convex programming techniques for LMIs. The design methodology is illustrated by an example of the balancing and swing-up of an inverted pendulum on a car.

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