Taylor Series Analysis of Multi-delay Systems

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ABSTRACT: The Taylor delayed operational matrix is extended to the analysis of multi-delay dynamic systems. A simple example is given to compare the exact solution and the solution obtained by the method shown in this paper. The result proves to be quite satisfactory.

I. Introduction

In recent years interest has been growing in the research of algebraic methods for analyzing and studying single delay or multi-delay problems of dynamic systems. However, efforts have primarily concentrated on the applications of orthogonal polynomials. Typical examples are Walsh functions (1, 2), block pulse functions (3), and shifted Legendre series (4).

These methods usually give rise to extremely complicated algorithms. Furthermore, more computing time and storage is needed when performing the digital simulation. In this work, we attempt to obtain a more efficient and convenient approach. The Taylor series approach is thus extended to the analysis of multi-delay dynamic systems.

In this presentation, the matrix, called Taylor Delayed Operational Matrix, together with the operational matrix of integration, is employed to translate the integral equation to a set of algebraic equations. A simple example is given to compare the exact solution and the solution obtained by the methods of this paper. The result is proven to be quite satisfactory.

II. Taylor Operational Matrix

A function \( y = y(t) \) which is analytic in the neighborhood of the point \( t_0 = 0 \) may be expanded into power series using the Maclaurin formula

\[
y(t) = \sum_{n=0}^{\infty} a_n f_n(t)
\]

where

\[
f_n(t) = t^n, \quad a_n = \frac{1}{n!} \left( \frac{d^n y(0)}{dt^n} \right).
\]

With a view to obtaining an approximate expression of the analytic function
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\( f(t) \) over the interval \((0, T)\) where \( f(t) \) may be assumed analytic, one can truncate the series (1) up to the \( m \)th term:

\[
y(t) \simeq \sum_{n=0}^{m-1} a_n f_n(t)
\]

or

\[
y(t) \simeq \mathbf{a}^T \mathbf{f}(t)
\]

where \( \mathbf{a}^T = [a_0, a_1, \ldots, a_{m-1}] \), \( \mathbf{f}^T(t) = [f_0(t), f_1(t), \ldots, f_{m-1}(t)] \).

The Taylor series basis functions satisfy the following recurrence relation:

\[
f_n(t) = t f_{n-1}(t).
\]

Also, one may easily show that

\[
\int_0^t f_n(\tau) \, d\tau = \frac{t^{n+1}}{n+1} = \frac{t}{n+1} f_n(t) = \frac{1}{n+1} f_{n+1}(t).
\]

By invoking Eqs (5) and (6), one can obtain an operational matrix of integration \( \mathbf{P} \) that has the following simple form:

\[
\mathbf{P} = \begin{bmatrix}
0 & 1 & 0 & 0 & \ldots & 0 & 0 \\
0 & 0 & 1/2 & 0 & \ldots & 0 & 0 \\
0 & 0 & 0 & 1/3 & \ldots & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & 0 & 0 & \ldots & 1/m-2 & 0 \\
0 & 0 & 0 & 0 & \ldots & 0 & 1/m-1 \\
0 & 0 & 0 & 0 & \ldots & 0 & 0 \\
\end{bmatrix}
\]

as introduced by Mouroutsos \textit{et al.} in 1985 (5). Using the property of the Taylor operational matrix, the following relationship is obtained:

\[
\int_0^t \mathbf{f}(\tau) \, d\tau \cong \mathbf{P} \mathbf{f}(t), \quad 0 \leq t < 1
\]

where \( \mathbf{f}(t) \) is the function basis vector, defined as

\[
\mathbf{f}^T(t) = [1, t, t^2, \ldots, t^{m-1}]
\]

assuming that we take \( m \) terms.

Next, the \textit{Taylor Delayed Operational Matrix} would be developed as follows:

Let the function basis \( f_0(t) \) be shifted along the time axis by \( k \). Mathematically, it is denoted as \( f_i(t-k), i = 0, 1, 2, \ldots, m-1 \). Thus, one can establish the following relationship:

\[
f_0(t-k) = 1 = f_0(t)
\]

\[
f_1(t-k) = t-k = -k+f_1(t)
\]
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\[ f_2(t-k) = (t-k)^2 = k^2 - 2kf_1(t) + f_2(t) \quad (11) \]
\[ f_3(t-k) = (t-k)^3 = -k^3 + 3k^2f_1(t) - 3k^3f_2(t) + f_3(t) \quad (12) \]
\[ \vdots \]
\[ f_{m-1}(t-k) = (t-k)^{m-1} = \sum_{i=0}^{m-1} \binom{m-1}{i} t^i (-k)^{m-1-i} \quad (13) \]

Eqs (9)–(12) could be written in matrix-vector form. That is,

\[
\mathbf{f}_m(t-k) = \begin{bmatrix}
1 & 0 \\
-k & 1 \\
k^2 & -2k \\
-k^3 & 3k^2 \\
\vdots & \vdots \\
(-k)^{m-1} \left( \begin{array}{c}
m-1 \\
0
\end{array} \right) & (-k)^{m-2} \left( \begin{array}{c}
m-1 \\
1
\end{array} \right)
\end{bmatrix}
\begin{bmatrix}
0 & 0 & \cdots & 0 \\
0 & 0 & \cdots & 0 \\
1 & 0 & \cdots & 0 \\
-3k & 1 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
(-k)^{-m+3} \left( \begin{array}{c}
m-1 \\
2
\end{array} \right) & (-k)^{-m+4} \left( \begin{array}{c}
m-1 \\
3
\end{array} \right) & \cdots & 1
\end{bmatrix} \begin{bmatrix}
f_0(t) \\
f_1(t) \\
f_2(t) \\
f_3(t) \\
\vdots \\
f_{m-1}(t)
\end{bmatrix}
\]

\[ = \mathbf{Q}_{m \times m}(k) \mathbf{f}_m(t) \quad (14) \]

or, equivalently,

\[ \mathbf{f}_m(t-k) = \mathbf{Q}_{m \times m}(k) \mathbf{f}_m(t) \quad (15) \]

where \( \mathbf{Q}_{m \times m}(k) \) is called the Taylor delayed operational matrix.

III. Multi-Delay Systems

Let us consider the multi-delay linear dynamic system,

\[ \mathbf{x}(t) = \mathbf{A} \mathbf{x}(t) + \sum_{j=1}^{\ell} \mathbf{B}_j \mathbf{x}(t-k_j) + \mathbf{D} \mathbf{u}(t) \quad (16a) \]
\[ \mathbf{x}(t) = \mathbf{F}(t) \quad \text{for} \quad t < 0 \quad (16b) \]
where \( k_1, k_2, \ldots, k_r \) are positive integers with \( k_1 \leq k_2 \leq \cdots \leq k_r \), and \( x(t) \in \mathbb{R}^n \); \( u(t) \in \mathbb{R}^p \).

Since the function \( F(t) \) is specified, the following data can be computed in advance, that is

\[
F(t-k_j) = V_j f_{\omega_0}(t)
\]

\[
\int_0^{\tau_j} F(t-k_j) \, dt = Z_i
\]

for \( j = 1, 2, \ldots, r \), where \( V_j \) is an \( n \times m \) matrix and this matrix is the coefficient of function \( F(t-k_j) \).

In this case, we make the assumption that

\[
k_1 < k_2 < \cdots < k_r.
\]

For \( 0 \leq t < 1 \), the solution of (16) can be formulated as follows:

By integration, (16) becomes

\[
x(t) = \int_0^t A x(\tau) \, d\tau + \sum_{j=1}^{r} B_j x(t-k_j) \, d\tau + \int_0^t D u(\tau) \, d\tau + x_0.
\]

Again, let \( x(t) \) and \( u(t) \) be represented as, respectively,

\[
x(t) \cong H f_{\omega_0}(t)
\]

and

\[
u(t) \cong G f_{\omega_0}(t)
\]

where \( H \) and \( G \) are \( n \times m \) and \( p \times m \) matrices, respectively. Then the solution of (19) can be obtained by using a method of partitioning it into several time sections. Each can be evaluated separately as follows:

For \( 0 \leq t < k_1 \),

\[
x(t) = \int_0^t A x(\tau) \, d\tau + \sum_{j=1}^{r} B_j F(t-k_j) \, d\tau + \int_0^t D u(\tau) \, d\tau + x_0
\]

putting (17) and (20) in (21), one may obtain

\[
H f_{\omega_0}(t) = A H f_{\omega_0}(t) + \sum_{j=1}^{r} B_j V_j f_{\omega_0}(t) + D G f_{\omega_0}(t) + E f_{\omega_0}(t)
\]

where \( P \) is called operational matrix of integration, and

\[
E = [x_0 \ 0 \ 0 \ \cdots \ 0]^{\text{trans}}.
\]

Therefore,
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\[ H = AHP + \sum_{j=1}^{r} B_j V_j P + DGP + E. \]  

(23)

Once \( H \) is determined from the above equation, \( x(t) \) would be found. For \( k_1 \leq t < k_2 \),

\[ x(t) = \int_{0}^{t} A x(\tau) \, d\tau + \int_{0}^{k_1} B_1 F(\tau - k_1) \, d\tau + \int_{0}^{k_1} B_1 x(\tau - k_1) U(\tau - k_1) \, d\tau \]

\[ + \int_{0}^{r} \sum_{j=2}^{r} B_j F(\tau - k_j) \, d\tau + \int_{0}^{r} D u(\tau) \, d\tau + x_0. \]  

(24)

Inserting (17), (18) and (20) into (24), one has

\[ H f_{(m)}(t) = AHP f_{(m)}(t) + B_1 Z_1 f_{(m)}(t) \]

\[ + B_1 H \int_{0}^{t} f_{(m)}(\tau - k_1) U(\tau - k_1) \, d\tau \]

\[ + \sum_{j=1}^{r} B_j V_j P f_{(m)}(t) + DGP f_{(m)}(t) + E f_{(m)}(t) \]

\[ = AHP f_{(m)}(t) + B_1 Z_1 f_{(m)}(t) + B_1 HPQ f_{(m)}(t) \]

\[ + \sum_{j=2}^{r} B_j V_j P f_{(m)}(t) + DGP f_{(m)}(t) + E f_{(m)}(t) \]

where

\[ Z_1 = [z_1, \begin{array}{cccc} 0 & 0 & \ldots & 0 \end{array}] \]

\((m-1)\) columns

Therefore,

\[ H = AHP + B_1 Z_1 + B_1 HPQ + \sum_{j=2}^{r} B_j V_j P + DGP + E. \]  

(25)

Equation (25) can be solved in a similar way as previously discussed. For a general solution, the \( x(t) \) can be evaluated from the following \( n \times m \) algebraic equations:

\[ H = AHP + \sum_{j=1}^{r} B_j Z_j + \sum_{j=1}^{r} B_j HPQ + \sum_{j=r+1}^{r} B_j V_j P + DGP + E \]  

(26)

for \( k_i \leq t < k_{i+1}, \quad i = 1, 2, \ldots, r-1 \), and

\[ H = AHP + \sum_{j=1}^{r} B_j Z_j + \sum_{j=1}^{r} B_j HPQ + DGP + E \]  

(27)

for \( k_r \leq t < 1 \). Thus, the solutions of (26) for \( i = 1, 2, \ldots, r-1 \) as well as the solutions of (23) and (27) form the solution of the multi-delay system.
IV. Numerical Example

Consider the delay system described by

\[ x(t) = x(t-0.35) + x(t-0.7) + 1 \]  \hspace{1cm} (28a)

\[ x(t) = 0 \hspace{0.5cm} \text{for} \hspace{0.5cm} t \leq 0. \]  \hspace{1cm} (28b)

Equation (28) may be solved by partitioning it into three sections, each evaluated separately as follows:

For, \(0 \leq t < 0.35\), and taking (23) becomes

\[ H = GP \]  \hspace{1cm} (29)

where

\[ H = [h_0 \ h_1 \ h_2 \ h_3] \]

\[ G = [1 \ 0 \ 0 \ 0] \]

and

\[ P = \begin{bmatrix}
0 & 1 & 0 & 0 \\
0 & 0 & 1/2 & 0 \\
0 & 0 & 0 & 1/3 \\
0 & 0 & 0 & 0
\end{bmatrix} \]

The solution of (26) is

\[ H = \begin{bmatrix} 0 & 1 & 0 & 0 \end{bmatrix} \]

that is \(x(t) = t\).

For \(0.35 \leq t < 0.7\), (26) becomes

\[ H = B_1 HPQ + DGP \]  \hspace{1cm} (30)

where

\[ Q_{(0.35)} = \begin{bmatrix}
1 & 0 & 0 & 0 \\
-7/20 & 1 & 0 & 0 \\
49/400 & -7/10 & 1 & 0 \\
-343/8000 & 147/400 & -21/20 & 1
\end{bmatrix} \]

\[ D = B_1 = 1. \]
By the solution of (30), one has

\[ H = \begin{bmatrix} 0.0328136 & 0.7916526 & 0.2932047 & 0.0977349 \end{bmatrix} \]

that is,

\[ x(t) = 0.328136 + 0.7916526t + 0.2932047t^2 + 0.0977349t^3. \]

For \( 0.7 \leq t < 1 \), (27) becomes

\[ H = B_1 HPQ_{1,4}(0.35) + B_2 HPQ_{1,4}(0.7) + DGP \]  \( \text{(31)} \)

where \( B_1 = B_2 = 1 \), where

\[ Q_{1,4}(0.7) = \begin{bmatrix} 1 & 1 & 0 & 0 \\ -7 & 1 & 0 & 0 \\ 10 & -7 & 1 & 0 \\ 49 & 5 & -7 & 1 \\ 100 & 100 & 10 & -21 \\ 1000 & 1000 & 10 & 1 \end{bmatrix} \]

Solving (31), one gets

\[ H = \begin{bmatrix} 0.0784747 & 0.6606534 & 0.3222699 & 0.2148466 \end{bmatrix} \]

that is,

\[ x(t) = 0.0784747 + 0.6606534t + 0.3222699t^2 + 0.2148466t^3. \]

Figure 1 gives a comparison of the Taylor approximate solution with the exact solution and the Walsh solution.

The exact solution is

\[ x(t) = \begin{cases} t, \text{ for } 0 \leq t < 0.35 \\ \frac{1}{2}(t - 0.35) + t, \text{ for } 0.35 \leq t < 0.7 \\ \frac{1}{2}(t - 0.35)^2 + \frac{1}{2}(t - 0.7)^2 + \frac{1}{2}(t - 0.7)^3 + t, \text{ for } 0.7 \leq t < 1. \end{cases} \]

The simulation result obtained is quite satisfactory.

**V. Conclusions**

In this work, the Taylor operation matrix together with the delayed operational matrix has been used to analyze multi-delay systems. A general formula was also developed. The advantages of this method over the others can be summarized as follows:

1. It is suitable for digital computation;
2. It needs less computation time and memory;
3. The number of terms in the Taylor expansion is independent of the values of delays;
4. The algorithms derived are much simpler than those of other methods;
5. The accuracy is better than that of the Walsh method.

*Vol. 236, No. 1, pp. 65-72, 1987*  
*Printed in Great Britain*
For further study, this method can also be extended to time-varying systems.

References