GA-based $H_2/H_\infty$ static output feedback design with average performance concept and techniques of family of polynomials

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A B S T R A C T

For continuous systems with polytopic uncertainties, the mixed $H_2/H_\infty$ robust output control problem is considered in this paper. The design objective is to seek a static output feedback controller that is able to stabilize the uncertain system and to minimize the mixed $H_2/H_\infty$ norm. To this end, a hybrid algorithm mixed by the technique of Hurwitz testing matrix of family of polynomial, average performance constraint and genetic algorithm (GA) is proposed. Due to the fact that the algorithm is based on GAs, we use a hierarchical structure to merge the conditions of Hurwitz testing matrix technique and average performance constraint into a fitness function which is called Hierarchical Fitness Function Structure (HFFS). By replacing fitness function with HFFS, any GAs can be applied to constitute the algorithm. Also, the proposed approach does not need any assumptions, so it is much more relaxed than that of Lyapunov-based design methods.

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1. Introduction

At present, a lot of analysis and synthesis tools of robustness for both continuous and discrete time-invariant linear systems have been well developed (Boyd, Ghouai, Feron, & Balakrishnan, 1994; Gahinet, Nemirovski, Laub, & Chilali, 1995; Skelton, 1998; Syrmos, Abdallah, Dorato, & Grigoriadis, 1997; Weinmann, 1992). Most robust control designs were derived from Lyapunov stability criterion. For the designs of state feedback control, the mixed $H_2/H_\infty$ robust control problems have been shown to be equivalent to the convex problems that can be solved by Linear Matrix Inequality (LMI) optimization (Boyd et al., 1994; Gahinet et al., 1995). Unfortunately, the designs of static output feedback control are not convex problems so that some hypotheses or assumptions (Leibfritz, 2001; Skelton, 1998) are necessary when converting Lyapunov-based stability conditions into LMI form. Nevertheless, these hypotheses or assumptions often lead to overly conservative design results especially for the system involving uncertainties.

In this paper, the design objective is to seek a static output feedback controller directly so that the mixed $H_2/H_\infty$ norm can be minimized. Therefore, this paper attempts to propose a hybrid algorithm, which is mixed by Hurwitz testing matrix technique in the field of the family of polynomials (Barmish, 1994; Weinmann, 1992), average performance (Boers, 2002) concept and genetic algorithms (GAs) (Bor-Sen & Yu-Min, 1998; Bor-Sen, Yu-Min, & Ching-Hsiang, 1995; Goldberg, 1989; Man, 1997). The Hurwitz testing matrix technique is used to relax the constraint on robust stability, and the average performance concept is used to improve the entire performance. The evaluation structure of the hybrid algorithm is based on GAs, which is used to seek the static output feedback gains such that the polytopic system is stabilized and the mixed $H_2/H_\infty$ norm is minimized. To this end, we use a hierarchical structure to merge the conditions of Hurwitz testing matrix technique and average performance constraint into a fitness function, which is called HFFS. The advantages of HFFS are that the procedures in computing fitness value do not need to deal with unused conditions or constraints and the fitness value is capable of indicating the infeasible condition or performance constraint. Therefore, HFFS can greatly improve the effect on computing fitness value. Based on HFFS, any GAs can be applied to implement the proposed approach. It is important to emphasize that the hybrid approach is much more relaxed than Lyapunov-based one because the hypothesis of common Lyapunov inequalities is unnecessary. Finally, the difference between the proposed idea and LMI-based design approaches is shown in the numerical example.

2. System descriptions and problem formulation

Consider a continuous-time polytopic model (Boers, 2002)

\[
\dot{x}(t) = A(\theta)x(t) + B_{u}(\theta)u(t) + B_{v}(\theta)v(t),
\]

\[
\Sigma(\theta) : \begin{cases}
x(t) &= C_{x}x(t), \\
y(t) &= C_{y}x(t) + D_{2}u(t), \\
z_{e}(t) &= C_{z}x(t) + D_{3}u(t), \\
z_{o}(t) &= C_{z}x(t) + D_{4}o(t) + D_{5}u(t),
\end{cases}
\]

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where $x(t) \in \mathbb{R}^n$ is the state, $u(t) \in \mathbb{R}^m$ is the control input, $\omega(t) \in \mathbb{R}^q$ is an external disturbance, $y(t) \in \mathbb{R}^p$ is the measured output, $z_1(t) \in \mathbb{R}^{m_1}$ and $z_2(t) \in \mathbb{R}^{m_2}$ are the output to be controlled for $H_2$ and $H_{\infty}$ performances, respectively. The parameters $A(\theta)$, $B_{uk}(\theta)$ and $B_{uk}(\theta)$ of (1) are assumed of the following structure:

$$A(\theta) = \sum_{k=1}^{r} \theta_k A_k, \quad B_{uk}(\theta) = \sum_{k=1}^{r} \theta_k B_{uk}, \quad B_{uk}(\theta) = \sum_{k=1}^{r} \theta_k B_{uk},$$  \hspace{1cm} (2)

where $[A_k, B_{uk}, B_{uk}]$ are constant matrix/vector. This means that the $k$th edge of polytopic model (1), $\theta = [\theta_1, \theta_2, \ldots, \theta_r] \in \mathbb{R}^r$ is the uncertain vector which belongs to the set

$$\Theta = \left\{ \theta \in \mathbb{R}^r : \sum_{k=1}^{r} \theta_k = 1, \quad \theta_k \geq 0, \quad k = 1, 2, \ldots, r \right\}.$$  

By substituting $u(t) = -Gy(t)$ into (1), one can obtain

$$\Sigma(\theta) := \begin{bmatrix} x(t) = A(\theta)x(t) + B_{uk}(\theta)\omega(t), \\
    z_1(t) = C_{uk}x(t), \\
    z_2(t) = C_x x(t) + D_{uk}\omega(t) \end{bmatrix},$$  

where $A(\theta)$, $C_{uk}$ and $C_x$ are defined in the following mapping:

$$A_\theta(\theta) \equiv \sum_{k=1}^{r} \theta_k (A_k - B_{uk}GC), \quad C_\theta \equiv (C_2 - D_{uk}GC), \quad \text{and} \quad C_x \equiv (C_x - D_{uk}GC).$$  

The design objective of this paper is to seek an acceptable $G$ such that:

1. the closed-loop system (3) is robustly stable;
2. the following trade-off criterion is minimized:

$$J(\Sigma(\theta)) = \phi \| H_{\infty}(s, \theta) \|_\infty^2 + \psi \| H_2(s, \theta) \|_2^2,$$  \hspace{1cm} (4)

where $\phi \geq 0$ and $\psi \geq 0$ are scale factors; $H_{\infty}(s, \theta)$ and $H_2(s, \theta)$ are the closed-loop transfer functions from $\omega$ to $z_1$ and $z_2$, respectively. $\| \cdot \|_2$ and $\| \cdot \|_\infty$ denote $H_2$ and $H_{\infty}$ norm, respectively.

To achieve the above objectives, one of the popular approaches is derived from Lyapunov stability criterion, which is shown as follows.

**Theorem 1.** The closed-loop system (3) is robustly stable with $\| H_{\infty}(s, \theta) \|_\infty \leq \gamma$ if there exist a symmetric matrix $X_\infty > 0$ and $G$ such that

$$\begin{bmatrix} \bar{A}_k X_\infty + X_\infty \bar{A}_k & * & * \\
  B_{uk}^T & -1 & * \\
  C_{uk} X_\infty & D_{uk} & -\gamma I \end{bmatrix} < 0, \quad \text{for } k = 1, 2, \ldots, r,$$  \hspace{1cm} (5)

where $\bar{A}_k = [A_k - B_{uk}GC]$; the notation $*$ is used to indicate terms that can be induced by symmetry.

**Remark 1.** It is clear that (5) involves the stability conditions (6).

Therefore, the mixed $H_2/H_{\infty}$ robust control problem shown in (4) can be represented as follows:

Minimize $\psi \phi^2 + \psi \mathrm{Trace}(Y)$ subject to (5), (7).

$$X = X_\infty > 0 \text{ and } Y \geq 0.$$  

It is well known that LMI-based approaches are based on the worst-case strategy, but none of information can indicate what kind of situation yields the worst-case result. Therefore, the computed performances could be overly conservative. Also, the hypothesis $X = X_\infty = X_2 > 0$ is necessary when solving $G$ with LMI optimization (Gahinet et al., 1995; Scherer et al., 1997). Because $G$ is located between $C_\theta$ and $X_\infty$, it is necessary to add some hypotheses or assumptions (Gahinet et al., 1995; Scherer et al., 1997; Skelton, 1998; Syrmos et al., 1997) that are used to convert (5)–(7) into LMI forms. Nevertheless, these constraints or hypotheses may lead to overly conservative results especially for the polytopic model involving many edges.

3. Design concept and preliminaries

To overcome the above disadvantages, this paper develops a new approach to design a robust $H_2/H_{\infty}$ static output feedback controller. That is, we directly employ the numerical algorithm to seek the static output feedback gains such that the closed-loop polytopic system is asymptotically stable with minimum $H_2$/$H_{\infty}$ performance to relax the stability constraints of Lyapunov-based designs, this paper employs the generalized edge theorem (Soh & Foo, 1989) and the Hurwitz testing matrix technique (Barmish, 1994) in the field of family of polynomials to examine the robust stability. It should be emphasized that the generalized edge theorem is much more relaxed than Lyapunov stability criterion because it directly examines whether all zeros of the family of polynomials lie on the left-half side of complex plane or not. To improve the system performances, this work adopts the average performance strategy rather than the worst-case one (Boers, 2002). From the above statements, it is seen that the key point is to find a method to combine these techniques. Before introducing the proposed approach, three preliminaries need to be briefly reviewed in advance, i.e., generalized edge theorem, Hurwitz testing matrix technique and average performance concept.

3.1. Generalized edge theorem and Hurwitz testing matrix

It is well known that the generalized edge theorem and Hurwitz testing matrix are designed for examining the stability of family of polynomials. When the feedback gains $G$ are given, it is easy to convert the closed-loop polytopic system (3) into a family of polynomials.

**Definition 1** (Barmish, 1994). Let $G$ be given. The closed-loop polytopic system $A(\theta)$ can be converted into the family of polynomials by using the following formulation.

$$\phi = \left\{ p(s, \theta) : p(s, \theta) = \sum_{k=1}^{r} \theta_k p_k(s) ; p_k(s) = \det(sI - \bar{A}_k) \right\},$$  \hspace{1cm} (9)

where $p_k(s) = a_{nk}s^n + a_{nk-1}s^{n-1} + \cdots + a_1 s + a_0 \forall a_{nk}$ is 1 and $\phi$ denotes a family of polynomials.

In this paper, the generalized edge theorem is employed to examine the robust stability of $\phi$, which is revised from the faults of Kharitonov’s theorem and the Edge Theorem. The main result is shown in the following lemma.
Lemma 1 (Soh and Foo, 1989; Foo and Soh, 1992). For the family of polynomials (9), it will exactly have k zeros in a simply connected region D in the complex plane if and only if:

1. the linear combination \( \Gamma_y \) has exactly k zeros in D,
\[
\Gamma_y \equiv \{ p_m(s) : \text{re} \{ p_m(s) \} = x \},
\]
where \( ij = 1,2,\ldots,r \) and \( i < j \).
2. The whole set D does not have any zero at the boundary contour of D.

Corollary 1. Lemma 1 implies that \( A_i(\theta) \) is said to be robustly stable if all zeros of the family of polynomials (9) lie on the left-half side of imaginary axis. To check all zeros of \( \mathcal{H} \) inside the left-half side of imaginary axis, Foo and Soh (1992) analyzed the distribution of zeros of \( \mathcal{H} \) by plotting the root loci for all \( \Gamma_y \). To identify the stability of each \( \Gamma_y \) without plotting root loci, this paper uses the Hurwitz testing matrix to seek unstable linear combinations for all \( \Gamma_y \).

Theorem 3. The root loci of all \( \Gamma_y \) lie on the left-half side of complex plane if and only if all polynomials of \( \mathcal{H} \) are strictly Hurwitz, and the following conditions are satisfied:
\[
\gamma(H_i^\top H_i) \neq (\infty, 0),
\]
where \( i,j = 1,2,\ldots,r \) and \( i < j \). \( \gamma_{\text{min}}(\cdot) \) and \( \gamma_{\text{max}}(\cdot) \) denote the minimum real part and eigenvalues of \( \cdot \), respectively. The matrix \( H_i \) or \( H_j \) is called the “Hurwitz testing matrix,” which is defined as follows:
\[
H_i = \begin{bmatrix}
\alpha_{i-1,j} & \alpha_{i-2,j} & \ldots & \alpha_{i-5,j} & \ldots & 0 \\
\alpha_{ij} & \alpha_{i-2,j} & \ldots & \alpha_{i-4,j} & \ldots & 0 \\
0 & \alpha_{ij} & \alpha_{i-3,j} & \ldots & \alpha_{i-5,j} & \ldots & 0 \\
0 & 0 & \alpha_{ij} & \ldots & \alpha_{i-2,j} & \ldots & 0 \\
0 & 0 & 0 & \ldots & \alpha_{ij} & \ldots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \ldots & \vdots \\
0 & 0 & 0 & \ldots & \alpha_{ij}
\end{bmatrix}
\]

Proof. Recall eigenvalue criterion in Weinmann (1992) & Barmish (1994). Consider an nth order polynomial \( p_i(s) \) which is strictly Hurwitz and an mth order polynomial \( p_j(s) \). Then the linear combination \( p_i(s) + \gamma p_j(s) \) is strictly Hurwitz \( \forall \gamma \in (\gamma_{\text{min}}, \gamma_{\text{max}}) \), where
\[
\gamma_{\text{min}} = \max\{ \gamma, 1/\lambda_{\text{min}}(-H_i^\top H_i) \}, \quad \gamma_{\text{max}} = \begin{cases}
\gamma_i, & 1/\lambda_{\text{max}}(-H_i^\top H_i) < 0, \\
\infty, & \text{otherwise},
\end{cases}
\]
\[
\gamma_{\text{max}} = 1/\lambda_{\text{max}}(-H_i^\top H_i),
\]
where \( H_i \) is called the “Hurwitz testing matrix,” which has been defined in (12); \( \lambda_{\text{max}}(\cdot) \) and \( \lambda_{\text{min}}(\cdot) \) are determined as follows:
\[
\lambda_{\text{max}}(\cdot) = \begin{cases}
\text{Re}_{\text{max}}(\cdot); & \text{if } \text{Re}_{\text{max}}(\cdot) > 0, \\
0; & \text{else},
\end{cases}
\]
\[
\lambda_{\text{min}}(\cdot) = \begin{cases}
\text{Re}_{\text{min}}(\cdot); & \text{if } \text{Re}_{\text{min}}(\cdot) < 0, \\
0; & \text{else},
\end{cases}
\]
where \( \text{Re}_{\text{max}}(\cdot) \) denotes the maximum real part of \( \cdot \).

Clearly, the linear combination \( f_i(s) + \gamma f_j(s) \) corresponds to \( \Gamma_{ij} \) when defining \( \gamma = \gamma(1 - \alpha) \). According to (10), one can infer that \( \gamma_{\text{min}} = 0 \) and \( \gamma_{\text{max}} = \infty \), which can be achieved if and only if \( \text{Re}_{\text{max}}(\cdot) < 0 \). Following this sense, it is easy to guarantee that all branches of root loci for \( \Gamma_{ij} \) lie on the complex left-half plane if and only if (10) holds.

3.2. Average performance concept

Definition 2 Boers, 2002. Let \( \mu \) be a probability measure defined on \( \Theta \), then average performance is defined as
\[
J(\Sigma_\theta) := \int_{\Theta} \langle \phi(\Sigma_\theta), 0 \rangle + \phi(\Sigma_\theta) \rangle^2 d\mu.
\]
In order to simplify the evaluation processes, (17) is reduced to a finite sum, i.e.,
\[
J(\Sigma_\theta) := \sum_{k=1}^\infty \phi(\Sigma_\theta(\theta_k)) + \phi(\Sigma_\theta(\theta_k))^2,
\]
where \( \theta_1, \theta_2, \ldots, \theta_n \in \Theta \).

Remark 2. It is convenient to compute \( ||H_n(\theta, \theta_k)||^0 \) and \( ||H_n(\theta, \theta_k)||^2 \) by use of the ready-made functions because each working point \( \theta_k \) corresponds to a single controlled system. Thus the average performance is defined as the expected value of the summation of the square of the \( H_n \) norm of the closed system over \( \Theta \).

By use of the above preliminaries, the objective is to find robust static output feedback gains such that the closed-loop system \( \Sigma(\theta) \) is robustly stable for all \( \theta \in \Theta \) and \( J(\Sigma_\theta) \) is minimized.

4. Main results

To increase the practicality, a standard GA to seek static output feedback gains \( G \) is used. By use of the above techniques, this section concentrates on developing the hybrid algorithm to seek \( G \) satisfying the design objective. For the sake of convenience, the algorithm is built on GAs, in which the fitness function is composed of some basic stability conditions and (11) and (18).

To this end, two important issues need to be clarified in advance, (i) how to use the above preliminaries to derive the stability conditions for \( \Sigma(\theta) \) and (ii) how to construct the algorithm with GAs.

4.1. Hierarchical robust stability conditions for \( \Sigma(\theta) \)

Theorem 4. The closed-loop system \( \Sigma(\theta) \) is said to be robustly stable if and only if the following conditions are satisfied in turn:
(i) \( \text{Re}_{\text{max}}(\lambda(A_k)) \leq -\psi < 0 \),
(ii) \( \text{Re}_{\text{max}}(\lambda(A_k)) \leq -\psi < 0 \),
(iii) \( d_1 > 1 \),
where \( A_k \equiv A_k - B_k C \), \( A_k \equiv \frac{1}{\lambda_{\text{max}}(-H_i^\top H_i)} \) is given, which is designed for satisfying the constraint on the decay rate. \( d_1 \) is called the percentage of robust stability defined as follows:
\[
d_1 = 1 + \frac{1 + \sum_{i=1}^n \sum_{j=1}^m \text{flag}(\Gamma_y)}{T_b},
\]
where \( T_b = \frac{t_{(r-1)}}{n} \) is the total number of the linear combinations \( \Gamma_y \); the flag(\( \Gamma_y \)) is defined as
\[
\text{flag}(\Gamma_y) = \begin{cases}
1; & \text{Re}_{\text{max}}(\lambda(H_i^\top H_i)) > 0, \\
0; & \text{else},
\end{cases}
\]

Proof. Theorem 3 has shown that all \( f_i \) must be strictly Hurwitz before performing (11). That is, (20) is used to guarantee that all eigenvalues of the closed-loop systems in all working points must lie on the left-half side of complex plane. It is sensible that the closed-loop nominal system must be stable before solving (20).
Therefore, (19) needs to be satisfied before dealing with (20). The definition of $d_r$ is inferred from Hurwitz testing matrix technique. In (23), flag($f_i(\theta)$) = 1 implies that all branches of the root locus of the polynomial $f_i(\theta)$ lies on the left-half side of the complex plane. Clearly, $R_c(h)$ is said to be robustly stable when flag($C_{ij}(\theta)$) = 1, $i, j = 1, 2, \ldots, r$, i.e., $d_i > 1$. From the above statements, one can note that the conditions (19)–(21) must be satisfied in turn.

### 4.2. GA-based robust controller seeking algorithm

According to the conditions of Theorem 4, the remaining work is to seek an optimal $G$ so that (19)–(21) can be satisfied and the cost function (18) can be minimized. In this paper, the genetic approach is applied to achieve this end due to the following reasons.

(i) In the field of control, many researchers are used to solve control problems with genetic algorithms (Abdel-Magid & Abido, 2003; Hatzikos, Hatonen, & Owens, 2004; Man, 1997). (ii) GAs can deal with various multi-objective problems without changing its evaluation structure. (iii) The convergent analysis of GAs has been proved in Suzuki (1995).

In this paper, the GA-based design flowchart for seeking $G$ is shown in Fig. 1. It shows that $G$ is regulated according to the value which is obtained by performing (19)–(21) and (18). To keep the feature of Theorem 4, we use a hierarchical structure to merge (19)–(21) and (18) into a fitness function which is called HFFS. By replacing the fitness function with HFFS, one can use arbitrary GAs to seek $G$, e.g. Binary Code GA (BCGA), Real Code GA (RCGA) and Hybrid GA (HGA).

When constituting HFFS, two steps are required. The first step is to convert (18)–(21) into sub-fitness functions, and the second step is to join these sub-fitness functions together with the hierarchical structure so that the final fitness value is singleton. By using the following formulation, (18)–(21) can be respectively converted into the sub-fitness function:

$$fitness_i = \kappa \times d_i, \quad i = 1, 2, 3, 4$$

where $fitness_i$ stands for the $i$th sub-fitness function; $\kappa$ is the scaling factor used to amplify the value $d_i$, and $d_i$ corresponds to the fulfillment of $i$th required condition, which is defined as follows:

$$d_1 = (\delta_1 - \psi - \text{Re}_{\text{max}}(\lambda(\mathcal{A}_n))) / \delta_1,$$

$$d_2 = (\delta_2 - \psi - \text{max}_{1 < k < r} \text{Re}_{\text{max}}(\lambda(\mathcal{A}_k))) / \delta_2,$$

$$d_3 = d_r,$$

By use of the following logic structure, the sub-fitness values obtained from (23)–(27) can be merged into a singleton

$$ fitness = (i-1) \times \kappa + fitness_i $$

where $fitness$ denotes the final fitness value of HFFS; $fitness_i$ stands for the computing result of $i$th sub-fitness function.

![Fig. 1. Illustration of proposed idea.](image-url)
The relationship between the fitness value of HFFS and the hierarchical robust stability conditions is exhibited in Fig. 2. Obviously, $R_c(h)$ is robustly stable when $\text{fitness} > 3\kappa$. The average performance (17) is close to the optimal result when $\text{fitness} > 3\kappa$ and fitness is immovable all the while.

**Remark 4.** According to the definition of HFFS, the proposed algorithm constituted by GAs is able to pick out the infeasible condition and to avoid computing the unused conditions of HFFS. Therefore, time in computing fitness function can be greatly reduced.

According to Fig. 1 and the definition of HFFS, the complete structure of the proposed algorithm is shown in Fig. 3. Clearly, Fig. 3 is very similar to the structure of GAs, and the only difference is to replace the fitness function of the evaluation block with HFFS. Therefore, details of the performing procedures are omitted. Table 1 shows the comparison between the proposed approach and previous ones.

5. Numerical example

Consider the two-mass-spring system (Wie & Bernstein, 1992) shown in Fig. 4, which is a genetic model of an uncertain dynamical system with rigid-body mode and one vibration mode. This system can be represented in state-space form as

$$
\begin{bmatrix}
    x_1(t) \\
    x_2(t) \\
    x_3(t) \\
    x_4(t)
\end{bmatrix} =
\begin{bmatrix}
    0 & 0 & 1 & 0 \\
    0 & 0 & 0 & 1 \\
    -k/m_1 & k/m_1 & 0 & 0 \\
    k/m_2 & -k/m_2 & 0 & 0
\end{bmatrix}
\begin{bmatrix}
    x_1(t) \\
    x_2(t) \\
    x_3(t) \\
    x_4(t)
\end{bmatrix}
+ \begin{bmatrix}
    0 \\
    0 \\
    1/m_1 \\
    0
\end{bmatrix} [\omega_1(t)]
+ \begin{bmatrix}
    0 \\
    0 \\
    1/m_1 \\
    0
\end{bmatrix} [\omega_2(t)] + \begin{bmatrix}
    0 \\
    0 \\
    1/m_1 \\
    0
\end{bmatrix} u(t),
$$

(29a)

![Fig. 3. Evaluation structure of the proposed algorithm.](image)

![Fig. 4. ACC benchmark system.](image)
\[
y(t) = C_\alpha x(t),
\]
\[
z(t) = C_\alpha x(t) + D_{\alpha u} u(t),
\]
\[
z_\alpha(t) = C_\alpha x(t) + D_{\alpha u} u(t),
\]
\[
u(t) = G_{ij} x(t),
\]
where \(C_\alpha = [0 \ 1 \ 0 \ 0], \ C_\alpha = [0 \ 0 \ 0 \ 1], \ D_{\alpha u} = D_{\alpha u} = 0; \ k_1,\ m_1\) and \(m_2\) denote the spring coefficient, mass of car 1 and mass of car 2, respectively. In this example, we do not know the exact value of \(k_1\) and \(m_2\), yet we know that the range of each variable is \(0.5 \leq k \leq 2, 0.9 \leq m_1 \leq 1.1\) and \(0.9 \leq m_2 \leq 1.1\). The polytopic model of (29a) is established by use of the low and upper bound of each variable so that the uncertain model (29a) can be converted into the polytopic model with eight edges \((r = 8)\) and the relation between the sub-model and the edge of polytopic model is shown in Table 2. For example, the first edge shown in Table 2 is obtained by substituting \(k = 0.5, \ m_1 = 0.9\) and \(m_2 = 0.9\) into (29a) so that we can obtain
\[
A_1 = \begin{bmatrix}
0 & 0 & 1 & 0 \\
0 & 1 & 0 & 0 \\
0.5/0.9 & 0.5/0.9 & 0 & 0 \\
0.5/0.9 & -0.5/0.9 & 0 & 0
\end{bmatrix},
\]
\[
B_{a1} = \begin{bmatrix}
0 \\
0 \\
1/0.9 \\
0
\end{bmatrix}, \quad B_{h1} = \begin{bmatrix}
0 \\
0 \\
1/0.9 \\
0
\end{bmatrix}.
\]

Following this sense, the sub-model of each edge of (29a) can be obtained. In this design, the purpose is to design the static output feedback gains \(G\) such that the cost function (17) defined by average performance concept can be minimized. Two design cases are considered in this example, i.e., the state feedback design and the static output feedback one.

5.1. Case 1: state feedback design \((C_f = I)\)

5.1.1. Pure LMI-based design

After solving the constraints of Remark 1 with LMI optimization, one can obtain that the bound on the worst-case performance is \(7.9759 \times 10^{-4}\), where \(\|H\|^2 = 1.0171 \times 10^{-4}\) and \(\|H\|^2 = 6.9588 \times 10^{-4}\). The controller achieving the performance is shown as follows:
\[
G_{LM} = 10^5 \times [0.0067 \ 4.2089 \ 0 \ 0.6681]
\]

Unfortunately, the feedback gains are so large to destroy the characteristics of the original system. It implies that the pure LMI-based design has to add some extra conditions to restrict the range of feedback gains. Nevertheless, these extra conditions could result in conservative results.

5.1.2. Proposed method

Now, the proposed idea is used to solve the state feedback gains subject to the following specifications: \(k = 10, \ \psi = 0.001\) and \(|g_{ij}| \leq 10\), where \(g_{ij}\) denotes the feedback gain in the \(i\)th row and \(j\)th column of \(G\). For the sake of convenience and simplicity, BCGA is applied to constitute the proposed algorithm, of which the specification is listed in Table 3. Because this paper focuses on the issue of controller design, the parameters of Table 3 are the recommended value and we do not further discuss how to select best parameters. The details about choosing these parameters can be found in Goldberg (1989). Because the evaluation structure of the proposed idea is similar to GAs, the following statements do not show the details of the seeking procedures and only focus on explaining the details of computing the fitness value of HFFS. Assume that a chromosome has been decoded as follows:
\[
G = [2.0117 \ \ -2.4414 \ \ 1.8750 \ \ -0.2148]
\]
By substituting (32) into (18) and (19), one can obtain
\[
\lambda(\overline{A}_s) = -0.8245 + 1.6401i, -0.1224 + 0.8767i \quad \text{and} \quad \max_{1 \leq j \leq 3} \text{Re} \lambda(\overline{A}_s) = -0.0191.
\]
Therefore, it is easy to infer \(f_{\text{fitness}} = 11.3817\) and \(f_{\text{fitness}} = 10.1843\). Even if the closed-loop systems in all working points are stable, the linear combination \(\Gamma_j\) has 14 unstable root loci so that fitness3 = 5.3571. Since fitness3 < \(k\), it is unnecessary to compute fitness4 and the fitness value of HFFS is fitness = 25.3571, which indicates the fact that the three conditions are infeasible. After performing the algorithm with 128 iterations, we have \(f_\Sigma = 4.0281\) and the robust state feedback controller is shown as follows:
\[
G_{BF} = [9.2188 \ \ -4.8438 \ \ 5.0781 \ \ 9.9219]
\]

To verify the robust stability, the root locus of the edge polynomials is shown in Fig. 5. According to Lemma 1, all possible eigenvalues of \(\Sigma_j\) are within the region which is made up of the edges branches of root loci. To exhibit the relationship between the bound on feedback gains and performances, we reassign \(|g_{ij}| \leq 80\), and then find \(f_\Sigma = 1.182\) and the following result after performing the algorithm with 169 iterations.
\[
G_{BF} = [35.6250 \ \ 8.7500 \ \ 9.3750 \ \ 78.7500].
\]

5.2. Case 2: static output feedback design

Now, the following statements will concentrate on the design of static output feedback control, in which the specification is defined as \(|g_{ij}| \leq 10\) and the output vector is given as follows:
\[
C_f = \begin{bmatrix}
1 & 0 & 1 & 0 \\
0 & 1 & 0 & -2
\end{bmatrix}
\]
After performing the algorithm with 143 iterations, we have \(f_\Sigma = 4.6618\) and the robust state output feedback controller is shown as follows:
\[
G_{BF} = [9.9219 \ \ -7.1094].
\]
This phenomenon is confirmed from the design of LMI-based approach. In this example, one can find that the proposed approach does not need to change the evaluation structure or add any hypotheses when dealing with robust static output feedback problems. Also, the proposed idea can avoid the disadvantages of high-gain control.

6. Conclusions

In this paper, the mixed $H_2/H_{\infty}$ control problem has been investigated. For continuous systems, a hybrid algorithm mixed by GAs, Hurwitz testing matrix technique and average performance strategy has been developed to seek a static output feedback controller that minimizes the average of $H_2$ norm and $H_{\infty}$ norm for all working points. Since the evaluation structure of the hybrid algorithm is based on GAs, one can use arbitrary GAs to seek the feedback gains after replacing the fitness function with HFFS. Based on HFFS, the stability conditions and the average performance constraint can be dealt with in sequence so that time in computing fitness value can be greatly reduced and infeasible condition can be indicated. By modifying conditions of the HFFS, various control problems can be solved without adding any constraints. Therefore, the proposed algorithm is much more relaxed than Lyapunov-based design methods.

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References